

# Phonon momentum and damping of mechanical resonators

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The concept of physical momentum associated to phonons in a crystal, complemented with some fundamental reasoning, implies measurable effects in crystals even at a macroscopic scale. We show that, in close analogy with the transfer of momentum in the kinetic theory of gases, physical momentum carried by phonons couples the thermal and the velocity field in a vibrating crystal. Therefore an heat flow applied to a vibrating crystal can sustain or damp the oscillation, depending on the interplay between the temperature and the velocity gradient. We derive the general equations of this effect and show that its experimental confirmation is within reach of current technology.

In a crystalline solid, a phonon of wave vector  $\mathbf{k}$  carries a "crystal momentum"  $\hbar\mathbf{k}$  that participates in the interaction with particles or other elementary excitations that similarly carry their own crystal momentum. In fact the ions of the crystal can be seen as scattering centers, and the waves emitted from each ion have constructive interference only if the total crystal momentum is conserved up to an arbitrary reciprocal lattice vector. Therefore crystal momentum is not related to a physical momentum eventually associated to phonons, as they describe the relative internal motion of the atoms, and are not associated with any linear momentum of the crystal as a whole. The limits of this description becomes evident when the translational invariance of the crystal is broken, as in a finite-size body, or in case of randomly distributed scattering centers. In this case the anharmonic terms couple the relative internal motion with the center of mass motion, and phonons carry both crystal momentum  $\hbar\mathbf{k}$  and the physical linear momentum  $\mathbf{p} = 3\gamma\hbar\mathbf{k}$  [1–3]. The connection between these two forms of momentum is a measure of the anharmonicity and is given by the Grüneisen parameter  $\gamma$  of the lattice, which measures the relation between the volume of a crystal lattice and its vibrational properties.

The intimate connection between phonon momentum and thermal expansion has been first pointed out by Brillouin [4], who showed that thermal expansion can be viewed as the reaction of the crystal lattice to the internal pressure due to phonons. More recently, an experiment has demonstrated that the collision of phonons can move a small cargo along a carbon nanotube [5]. The concept of corpuscular phonon allows also a theoretical description of some dynamical phenomena in thermoelectricity, such as dynamical thermal expansion in thin optical windows [6], and is at the basis of the phonon pumping mechanism recently proposed for the cooling of nanodevices [7]. Here we show that phonon momentum transfer couples the thermal and the velocity field in a vibrating crystal and we predict measurable effects

in crystals even at a macroscopic scale.

This issue is relevant for a thorough understanding of oscillating systems of every size and for setting the limits to the performances of many instruments and devices. In fact resonant mechanical systems serve as a basis for a number of sensors and fundamental studies [8–11]. Their size covers orders of magnitude, from micrometers to some meters, and their motion is detected, by electrical or optical readouts, with high sensitivity, sometimes limited only by the uncertainty principle. In these instruments a low damping of the resonant motion is often crucial to achieve the desired performances, as in the case of gravitational wave detectors [12] or micro-mechanical resonators [13]. Here the laser beam used in optical readout usually heats some parts of the resonator: thermal gradients up to 10 K are expected in some components of the future cryogenic gravitational wave detectors [14], while gradients of about 1 K are currently applied in cryogenic micro-resonators with optical readout [11].

As discussed above, a physical linear momentum  $\mathbf{p} = 3\gamma_{\mathbf{k}s}\hbar\mathbf{k}$  is associated to a phonon of the branch  $s$ , where  $\mathbf{i}_{\mathbf{k}}$  determines the direction of propagation of the phonon and  $\gamma_{\mathbf{k}s}$  is the Grüneisen parameter of the normal mode. As long as we deal with crystals at temperatures much smaller than the Debye temperature (for instance  $\Theta_D = 645$  K for silicon), we can use the simple dispersion relation  $\omega_s(\mathbf{k}) = ck$  and write  $\mathbf{p} = 3\gamma_{\mathbf{k}s}\hbar\frac{\omega_k}{c}\mathbf{i}_{\mathbf{k}}$ . Accordingly, an equivalent mass  $m_k = 3\gamma_{\mathbf{k}s}\hbar\frac{\omega_k}{c^2}$ , arising from the coupling of the oscillatory motion of the atoms with the center of mass of the crystal, is associated to a phonon moving with velocity  $c\mathbf{i}_{\mathbf{k}}$ . In case of a phonon in a moving crystal, this relation must be complemented with a term depending by the motion of the center of mass. If we push forward the analogy between a phonon and a particle, it is natural to assign a momentum:

$$\mathbf{p} = 3\gamma_{\mathbf{k}s}\hbar\frac{\omega_k}{c^2}(c\mathbf{i}_{\mathbf{k}} + \mathbf{v}_Q) \quad (1)$$

if the phonon is in a crystal moving with velocity  $\mathbf{v}_Q$  in respect to an observer at rest. This relation transforms

the physical momentum between reference systems in relative motion, considering that the velocity of the phonon relative to the crystal is constant. If the phonon is traveling in a crystal having parts moving with different velocities, we expect that it will couple in sequence with the parts of the crystal crossed by its trajectory, changing its frequency and/or direction as it moves. As a result of this interaction, the phonon will actually transfer momentum across the crystal. Even if this issue deserve dedicated molecular dynamics studies, as a first approach we assume that the phonon transfers to the crystal its energy and momentum when suffering a collision, starting a new path with the frequency and momentum corresponding to the local temperature and velocity.

To understand the physical meaning of these assumptions we propose a simplified model in Figure 1. An oscillating cantilever beam is heated from its free end, so that both the temperature and the amplitude of the velocity increase along the  $x$ -axis. We consider three thin sections of thickness  $\Delta x \ll l_f$ , say  $Q_j$  with  $j = 1 - 3$ , at intervals of length  $l_f$  along the cantilever, with  $l_f$  the Mean Free Path (MFP) of the phonons. Due to the temperature dependence of the phonon density, a larger number of phonons and with higher energy are emitted from the section  $Q_3$  in respect to those emitted from  $Q_1$ . Following the dominant phonon approximation [15], we assume that the whole thermal energy in a section can be ascribed to phonons of a single frequency  $\bar{\omega}$ . Then, in the time  $dt$ , we have  $(\bar{n}_1, \bar{n}_2, \bar{n}_3)$  phonons with frequency  $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$  starting respectively from  $(Q_1, Q_2, Q_3)$ , with  $\bar{n}_3 > \bar{n}_2 > \bar{n}_1$  and  $\bar{\omega}_3 > \bar{\omega}_2 > \bar{\omega}_1$  (Figure 1c). Now we consider the balance of the momentum  $p_z$  directed along the  $z$ -axis, at the central section  $Q_2$  when the cantilever goes through a zero crossing. We assume that all phonons starting from  $Q_1$  and  $Q_3$ , travel for a length  $l_f$  and then suffer a collision with the lattice in  $Q_2$ . According to Eq. (1), phonons carry a momentum proportional to  $\bar{\omega}_j v_j$ , then:

$$\frac{dp_z}{dt} \propto \frac{1}{2} \bar{n}_1 \bar{\omega}_1 v_1 + \frac{1}{2} \bar{n}_3 \bar{\omega}_3 v_3 - \bar{n}_2 \bar{\omega}_2 v_2 \quad (2)$$

where we take in account that only one half of the phonons starting in the  $Q_1, Q_3$  are emitted towards  $Q_2$ . It is evident that, in a linear approximation of the velocity  $v_2 = (v_1 + v_3)/2$ , we have  $p_z = 0$  in the isothermal case, when phonon density and frequencies do not changes along the cantilever. In this case the excess momentum coming from  $Q_3$  is compensated by a smaller momentum from  $Q_2$ . On the contrary a thermal gradient changes the number and the frequency of phonons coming from  $Q_1$  and  $Q_2$  and we easily obtain:

$$\frac{dp_z}{dt} \propto \frac{\partial(\bar{n}\bar{\omega})}{\partial x} \frac{\partial v}{\partial x} l_f^2 \quad (3)$$

showing that within this framework a combination of thermal and velocity gradients can transfer physical momentum along the cantilever.

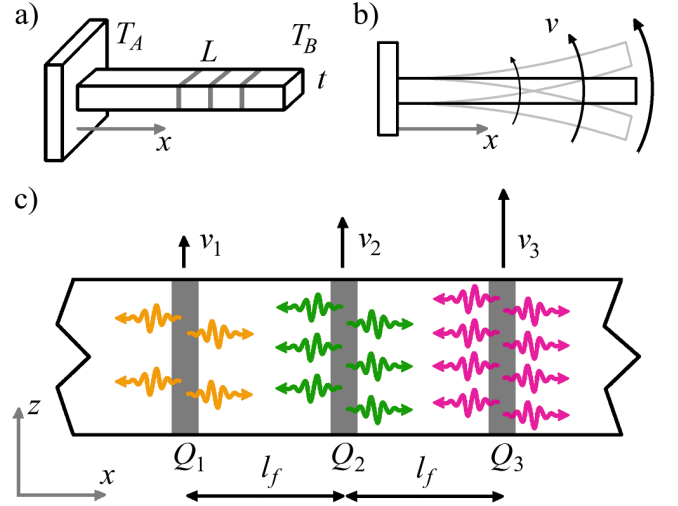


FIG. 1: (Color online) a) Heated cantilever beam, of length  $L$  and thickness  $t$ . The temperature increases along the  $x$ -axis from  $T_A$  to  $T_B$ . Three thin sections  $Q_k$ , at intervals of length  $l_f$ , are shown in gray. b) The cantilever oscillates in its first bending resonance mode. When it goes through a zero crossing, the local velocity is directed along the  $z$ -axis and its amplitude increases along the  $x$ -axis. c) Detail of sections  $Q_k$ . The density and the frequency of the phonons increase from left to right, proportionally to the local temperature. As phonons travel for a distance  $l_f$ , one half of the phonons starting from  $Q_1$  and  $Q_3$  reach the central section  $Q_2$ .

To evaluate the possible implications of this effect on the dynamics of a resonator, we have to put Eq. (3) on a more solid ground, in close analogy with the theory of transfer of momentum in the kinetic theory of gases [16]. We consider the point  $P \equiv (x, y, z)$  in a crystal, where the temperature is given by a scalar field  $T(x, y, z)$  and the velocity by a vector field  $\mathbf{v}(x, y, z)$ . We assume that a steady state has been reached and evaluate the balance of momentum in a small sphere  $\Lambda$  of radius  $d\lambda$  centered at  $P$ . We place a polar coordinate system at  $P$ , and consider a small volume  $d\sigma = r^2 \sin \theta dr d\theta d\phi$  at the point  $\mathbf{r} \equiv (r, \theta, \phi)$ , say  $Q$ , in this coordinate system (see Fig. 1 of the supplementary material [18]). Both volumes are small in the sense that their characteristic size is much smaller than the MFP  $l_f$ . For each phonon branch  $s$ , in  $d\sigma$  are  $n_s(T_Q, \mathbf{k})d\sigma$  phonons with wave vector  $\mathbf{k}$ , where  $T_Q$  is the value of the temperature at the point  $Q$ . Each phonon starts a free path  $c/l_f$  times per second, where  $c$  is the group velocity, therefore a number of  $\frac{n_s(T_Q, \mathbf{k})c}{l_f}d\sigma dt$  phonons will leave the volume  $d\sigma$  in the time  $dt$ . Of these, the number in a cone of base  $\pi(d\lambda)^2$  will have paths directed towards the small sphere if all directions are equally probable. That is,  $\frac{\pi(d\lambda)^2}{4\pi r^2} n_s(T_Q, \mathbf{k}) \frac{c}{l_f} d\sigma dt$  phonons will leave headed for the sphere along  $\mathbf{r}$ . Of these, a fraction only will suffer a collision within the sphere  $\Lambda$ : if we consider for instance only phonons directed exactly along  $-\mathbf{r}$ , the frac-

tion  $e^{-(r-d\lambda)/l_f}$  will succeed in reaching the sphere at  $P$ , while the fraction  $e^{-(r+d\lambda)/l_f}$  will cross it without impact. As a result the number of collisions within  $\Lambda$  is  $\frac{2}{3}e^{-r/l_f}\frac{2d\lambda}{l_f}$ , where the factor  $\frac{2}{3}$  accounts for the reduction of the length of the path on the borders of the sphere. Hence the number of phonons with wave vector  $\mathbf{k}$  starting in  $d\sigma$  and colliding within the sphere  $\Lambda$  is:

$$dN = \frac{4}{3}\pi(d\lambda)^3 n_s(T_Q, \mathbf{k}) \frac{c}{4\pi r^2 l_f^2} e^{-r/l_f} d\sigma dt \quad (4)$$

The first term is the volume of the sphere  $\Lambda$ , that we approximate as a cubic volume element  $dV = dx dy dz$  to allow the integration over the crystal [17]. The collision rate at  $P$  is then:

$$\frac{dN}{dt} = n_s(T_Q, \mathbf{k}) \frac{c}{4\pi r^2 l_f^2} e^{-r/l_f} d\sigma dV \quad (5)$$

The components of the total rate of the momentum entering the element  $dV$  are obtained by integrating the product of equation (5) and (1). Here  $\mathbf{k}$  varies over the full phonon spectrum and the volume  $d\sigma$ , placed at the source point  $Q$ , over the volume of the body  $V_B$ :

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= dV \int_{V_B} d\sigma \left[ \sum_{\mathbf{k}s} \gamma_{\mathbf{k}s} \hbar \omega_{\mathbf{k}} n_s(T_Q, \mathbf{k}) \right] \times \\ &\times \frac{3}{4\pi r^2 l_f^2 c} e^{-r/l_f} (-c \mathbf{i}_r + \mathbf{v}_Q) \end{aligned} \quad (6)$$

where the direction of propagation of the phonons has been approximated as  $\mathbf{i}_{\mathbf{k}} \simeq -\mathbf{i}_r$ . This equation can be evaluated after some preliminary consideration. First, the volume integral can be extended over all space, because only a sphere of radius comparable with a few mean free path actually contributes to the volume integral, due to the exponential term  $e^{-r/l_f}$ . Second, we note that  $\hbar \omega_{\mathbf{k}} n_s(T, \mathbf{k}) = \int_0^T c_{vs}(\mathbf{k}) dT$ , where  $c_{vs}(\mathbf{k})$  is the contribution of the mode  $(\mathbf{k}s)$  to the specific heat  $C_v = \sum_{\mathbf{k}s} c_{vs}(\mathbf{k})$ . Then the sum over the phonon spectrum is  $\sum_{\mathbf{k}s} \gamma_{\mathbf{k}s} \int_0^T c_{vs}(\mathbf{k}) dT = \int_0^T \gamma(T) C_v dT$ , where  $\gamma(T) = \sum_{\mathbf{k}s} \gamma_{\mathbf{k}s} c_{vs}(\mathbf{k}) / C_v$  is the overall Grüneisen parameter. In general this integral has a complex functional dependence on the temperature [18], as  $\gamma(T)$  may change in a complicated way. On the other hand at very low temperatures (below 10 K) we have  $\int_0^{T_Q} \gamma(T) C_v dT \simeq \gamma_0 \frac{B_v}{4} T_Q^4$ , with  $B_v$  determined by the relation  $C_v \simeq B_v T^3$  and  $\gamma_0 = \lim_{T \rightarrow 0} \gamma(T)$ . At intermediate temperatures (30-70 K for silicon) this relation remains a good approximation with  $\gamma_0$  an effective Grüneisen parameter [18]. We then expand  $\mathbf{v}_Q$  and  $T_Q^4$  to I order in  $\mathbf{r}$  around the point  $P$ , and obtain for the components of  $\frac{d\mathbf{p}}{dt}$ :

$$\begin{aligned} \frac{dp_\alpha}{dt} &\simeq dV \gamma_0 \frac{B_v}{4} \times \\ &\times \left( \frac{3T_P^4}{l_f c} v_{P\alpha} - 4T_P^3 \frac{\partial T_P}{\partial \alpha} + \frac{8T_P^3 l_f}{c} \nabla T_P \cdot \nabla v_{P\alpha} \right) \end{aligned} \quad (7)$$

where  $\alpha = x, y, z$  and all derivatives are evaluated at the point  $P$ . The details of the calculation, with a discussion of II order terms not shown here, can be found in the supplementary material [18].

To complete the momentum budget we must consider the rate of momentum loss due to phonons starting a new free path in the element  $dV$ . For each direction  $(\theta, \phi)$  a number of  $dV n_s(T_P, \mathbf{k}) \frac{c}{l_f} \frac{\sin \theta d\theta d\phi}{4\pi} dt$  phonons will leave the element in the time  $dt$ , each carrying away the momentum given by Eq. (1). It is straightforward to show that this contribution, when integrated over the phonon spectrum and all directions, compensates the first term (thermal equilibrium term) of Eq. (7).

The second term (phonon pressure term) is proportional to the temperature gradient and describes the balance of momentum along the direction of propagation  $\mathbf{i}_{\mathbf{k}}$ . In fact, the distribution of phonons arriving from the hot part is shifted toward higher frequency, therefore the resultant momentum rate is directed against the temperature gradient. The force associated to this momentum rate is supported by the crystal through the elasticity modulus and induces a nonuniform expansion across the volume of the solid body. Hence our model extends to the case of a thermal gradient the well known connection between phonon momentum and thermal expansion [4, 19].

The third term (damping term) is proportional to the gradients of the temperature and of the velocity fields, and can have striking effects in the case of an oscillating crystal with a thermal gradient. In fact the rate of momentum transferred from the phonon gas to the crystal is equivalent to a force  $\mathbf{f}_P = \frac{d\mathbf{p}}{dt}$ . If the point  $P$  in the time  $dt$  changes its position by  $d\mathbf{x}_P$ , the work done by this force is  $\mathbf{f}_P \cdot d\mathbf{x}_P$ . This work can be easily evaluated if the system is oscillating with angular frequency  $\omega_0$ . In this case the components of the displacement field are  $u_\alpha(x, y, z, t) = G_\alpha f_\alpha(x, y, z) \sin(\omega_0 t)$ , where  $G_\alpha$  is the magnitude of the amplitude oscillation and  $f_\alpha(x, y, z)$  the displacement function. The velocity field and its gradient are respectively  $v_\alpha(x, y, z, t) = \omega_0 G_\alpha f_\alpha(x, y, z) \cos(\omega_0 t)$  and  $\nabla v_\alpha = \omega_0 G_\alpha \nabla f_\alpha \cos(\omega_0 t)$ . If we consider that the force  $\mathbf{f}_P$  is much smaller than the elastic forces at the origin of the oscillatory behavior, we can safely assume that it does not alter in a significant manner the displacement field  $u_\alpha(x, y, z, t)$ . Then the work done in an oscillation cycle is simply  $\delta L_P = \int_0^{2\pi/\omega_0} \mathbf{f}_P \cdot \mathbf{v}_P dt$ , and this local contribution can be integrated over the volume of the crystal to obtain the total work:

$$L_{Ph} = \frac{2\pi\gamma_0 l_f B_v}{c} \omega_0 G_u^2 \sum_{\alpha} \int_V (T^3 (\nabla T \cdot \nabla f_\alpha) f_\alpha) dV \quad (8)$$

This equation is the main result of this paper and must be carefully analyzed to understand the possible implications on the dynamics of the resonator.

We note first that the work  $L_{Ph}$  is proportional to the

squared amplitude of the oscillation, just like the energy losses due to internal friction in the resonator. On the other hand it can be positive or negative, depending on the interplay between velocity and temperature gradients, therefore the superimposed heat flow can sustain or damp the oscillation of the body. To compare more easily this effect with the usual dissipative phenomena in resonating bodies, we define the loss angle due the phonon momentum transport as  $\phi_{ph} = -\frac{1}{2\pi} \frac{L_{Ph}}{W}$ , where  $W = \frac{1}{2} \rho G_u^2 \omega_0^2 \sum_{\alpha} \int_V f_{\alpha}^2 dV$  is the total energy stored in the resonator. From Eq. (8) we have:

$$\phi_{ph} = -\frac{2\gamma_0 l_f B_v}{\rho \omega_0 c} \frac{\sum_{\alpha} \int_V (T^3 (\nabla T \cdot \nabla f_{\alpha}) f_{\alpha}) dV}{\sum_{\alpha} \int_V f_{\alpha}^2 dV} \quad (9)$$

where positive values of  $\phi_{ph}$  mean a damping of the oscillation. Besides this phononic contribution, the energy loss in a resonator results from the contribution of several dissipation mechanism, that collectively determine the background dissipation  $\Delta W_{bg}$  and the background loss angle  $\phi_{bg} = \frac{1}{2\pi} \frac{\Delta W_{bg}}{W}$ . The most common dissipation mechanism are thermoelastic and structural loss in the resonator and energy loss through the support. In this respect, we note that silicon resonators based on cantilevered beams have demonstrated a background loss as low as  $10^{-6}$  at cryogenic temperatures, with a reproducibility of about 20% [24].

For the cantilever of Fig. 1 the temperature profile is  $T(x) = (T_B - T_A)x/L + T_A$  and the displacement is fully described by the the component  $f_z$ . We use the variable  $\xi = x/L$ ,  $\xi \in (0, 1)$ , and write the modal shapes as:

$$f_z(\xi) = [\sin(\beta) - \sinh(\beta)] [\sin(\beta\xi) - \sinh(\beta\xi)] + [\cos(\beta) + \cosh(\beta)] [\cos(\beta\xi) - \cosh(\beta\xi)] \quad (10)$$

where  $\beta = 1.875$  for the first resonant mode [20]. The frequency of the oscillation is obtained as  $\omega^2 = \frac{\beta^2 t}{L^2} \sqrt{\frac{Y}{12\rho}}$ , where  $Y$  and  $\rho$  are respectively Young modulus and density of the crystal. In Fig. 2 we show the expected damping for the lowest frequency mode of a silicon cantilever ( $Y = 169$  GPa,  $\rho = 2329$  kg/m<sup>3</sup>) of thickness  $100 \mu\text{m}$  and lenght  $50$  mm. We use the average MFP from measurements of thermal conductivity accumulation distribution [21], and  $\gamma(T)$  from thermoelastic measurements [22, 23]. In this case Eq. (9) cannot be used below  $60$  K, as the MFP would become larger than the thickness of the cantilever and boundary scattering dominate the transport.

Now we can trace out the main features of the theory and evaluate its experimental relevance. At first we note that, with a relative gradient of 10%, the magnitude of  $\phi_{ph}$  can be well above the typical background loss in this systems [24]. Second, according to the sign of  $\gamma(T)$  and of the gradient, the additional loss  $\phi_{ph}$  can augment or reduce the overall damping of the resonator. Therefore with the application of a thermal gradient it is possible to reach a regime where the dynamics of the resonator

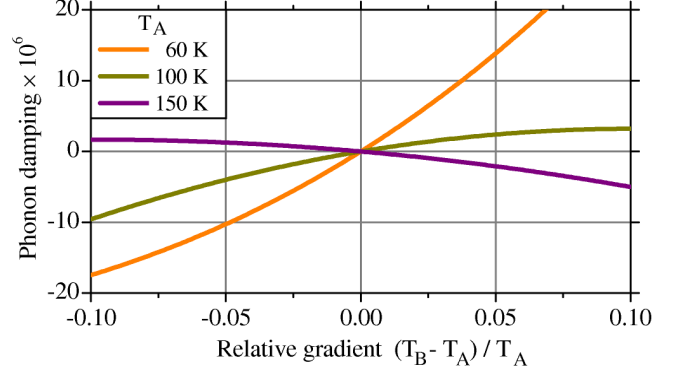


FIG. 2: (Color online) Expected damping due to the phonon momentum transport for the first bending resonance mode (55 hz) of the silicon cantilever. The curve at 60 K is obtained in the low temperature approximation (Eq. (9)) with  $\gamma_0 = -0.8$ , while curves at 100 K and 150 K are evaluated by taking in account the temperature dependence of the Grüneisen parameter [18]. A positive thermal gradient can be obtained by heating the free end of the beam. Then, depending on the base temperature, the additional loss  $\phi_{ph}$  can augment or reduce the background damping of the resonator. Typical background loss for a silicon cantilever of this size is  $\phi_{bg} \simeq 10^{-6}$ .

is dominated by the phononic damping. In this case, the loss angle can be much higher than expected background, spoiling off the performances of the device, or could be considerably reduced, possibly driving the system towards region of dynamic instability. From another perspective this effect could allow the design of micro-resonators exploiting the transport of phonon momentum as a power source, drawing energy from the thermal gradient to sustain their oscillatory motion. We point out that phonon momentum effects turn out to be accessible in a macroscopic system thanks to the extreme energy sensitivity of low loss resonant systems, while a wider variety of mechanical effects can be observed at the nanoscale [5].

We have extended the phonon momentum concept to the case of an oscillating crystal, and within this framework we have evaluated the dynamic behavior of an arbitrarily shaped resonator with an applied thermal gradient. In case of a cantilever we have found a number of features that are at experimental reach and could have a significant role in the performances of future devices for precision measurements. For this reason we hope that these ideas provoke new experimental and theoretical efforts aimed toward understanding the effect of phonon momentum on dissipative processes in resonators.

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**Supplemental material to:**  
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This PDF file includes the following material:

- Coordinate system (Figure 1)
- Temperature integral: approximation for silicon (Figure 2)
- Volume integral: first and second order approximation
- Volume integral: medium temperature approximation for silicon

# COORDINATE SYSTEM

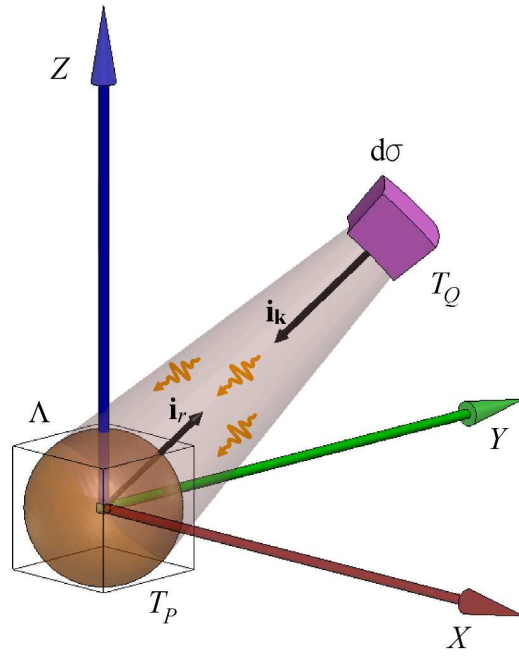


FIG. 1: Coordinate system used in the volume integration.

## TEMPERATURE INTEGRAL: APPROXIMATION FOR SILICON

The sum over the phonon spectrum in Equation (6) of the main text reduces to:

$$F(T_Q) \equiv \sum_{\mathbf{k}s} \gamma_{\mathbf{k}s} \int_0^{T_Q} c_{vs}(\mathbf{k}) = \int_0^{T_Q} \gamma(T) C_v dT \quad (1)$$

where  $\gamma(T) = \sum_{\mathbf{k}s} \gamma_{\mathbf{k}s} c_{vs}(\mathbf{k}) / C_v$  is the overall Grüneisen parameter and may change in a complicated way with the temperature. For instance in silicon  $\gamma(T)$  takes negative values between 19 K and 124 K. On the other hand the strong temperature dependence of the specific heat  $C_v$  can make negligible the low temperatures contribution to the integral.

In the case of silicon at very low temperatures (below 10 K) we have:

$$F_{VLT}(T_Q) \simeq \gamma_0 \frac{B_v}{4} T_Q^4 \quad (2)$$

with  $B_v = 0.591 \text{ J}/(\text{m}^3 \text{K}^4)$  determined by the relation  $C_v \simeq B_v T^3$  and  $\gamma_0 \simeq 0.4$  [1]. As shown in Figure 2c, this relation remains a good approximation in the range (30-70 K) if we use an effective Grüneisen parameter  $\gamma_0 \simeq 0.8$ .

In the medium temperature range (70-200) K, the temperature dependence of  $\gamma(T)$  changes the functional form of  $F(T)$ , that now is well approximated by a quadratic functions:

$$F_{MT}(T_Q) = F_0 + F_1 T_Q + F_2 T_Q^2 \quad (3)$$

with  $F_0 = 2.85 \times 10^7$ ,  $F_1 = -6.17 \times 10^5$  and  $F_2 = 2.57 \times 10^3$ .

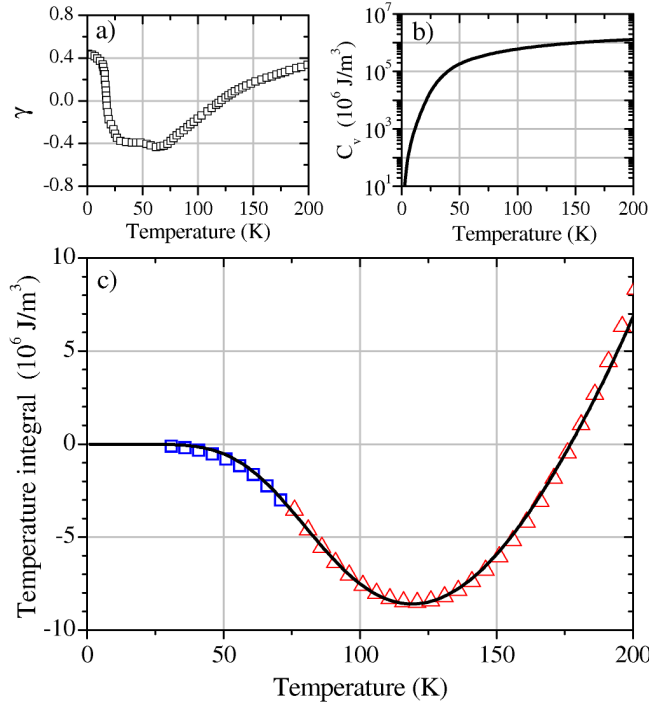


FIG. 2: Evaluation of the temperature integral  $F(T) = \int_0^{T_Q} \gamma(T) C_v dT$  for the case of silicon. a) Temperature dependence of the overall Grüneisen parameter  $\gamma(T)$  [2]; note that  $\gamma(T)$  is negative from 18 K to 124 K, where silicon has a negative coefficient of thermal expansion. b) Specific heat  $C_v$  of silicon [3]. c) Solid line: numerical evaluation of  $F(T)$ . Hollow squares: points from the low temperature approximation Eq. (2) but with an effective Grüneisen parameter  $\gamma_0 \simeq -0.8$ . Hollow triangles: points from the medium temperature quadratic approximation Eq. (3).



## VOLUME INTEGRAL: FIRST AND SECOND ORDER APPROXIMATION

From Equation (6) of the main text, after the average over the phonon modes, we write the rate of momentum entering the volume element  $dV$  as:

$$\frac{d\mathbf{p}}{dt} = dV \frac{3}{4\pi l_f^2 c} \int_{V_B} d\sigma F(T_Q) \frac{1}{r^2} e^{-r/l_f} (-c\mathbf{i}_r + \mathbf{v}_Q) \quad (4)$$

In the very low temperature case we have  $F(T_Q) = F_{VLT}(T_Q) \simeq \gamma_0 \frac{B_v}{4} T_Q^4$ , with  $\gamma_0 = \lim_{T \rightarrow 0} \gamma(T)$ . As shown in the Figure 2, this relation remains valid at higher temperatures with  $\gamma_0$  an effective Grüneisen parameter. Therefore Equation (4) becomes:

$$\frac{d\mathbf{p}}{dt} = \frac{B_v}{4} \frac{3\gamma_0}{4\pi l_f^2 c} dV \int_{V_B} d\sigma T_Q^4 \frac{1}{r^2} e^{-r/l_f} (-c\mathbf{i}_r + \mathbf{v}_Q) \quad (5)$$

With reference to the coordinate system shown in Figure 1, we expand  $\mathbf{v}_Q$  and  $T_Q^4$  up to II order in  $\mathbf{r}$ :

$$v_{Q\alpha} = v_{P\alpha} + \mathbf{r} \cdot \nabla v_{P\alpha} + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 v_{P\alpha} \quad (6)$$

$$T_Q^4 = T_P^4 + \mathbf{r} \cdot \nabla T_P^4 + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 T_P^4 \quad (7)$$

where the derivatives are evaluated at the point  $P$ , corresponding to  $\mathbf{r} = 0$ . The volume element is  $d\sigma = r^2 \sin \theta dr d\theta d\phi$ , therefore the volume integral for the component  $\alpha$ , with  $\alpha = x, y, z$ , is:

$$\int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta e^{-r/l_f} \left( T_P^4 + \mathbf{r} \cdot \nabla T_P^4 + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 T_P^4 \right) \left( -c\mathbf{i}_r \cdot \mathbf{i}_\alpha + v_{P\alpha} + \mathbf{r} \cdot \nabla v_{P\alpha} + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 v_{P\alpha} \right) \quad (8)$$

where we have extended the volume integral over all space, because only a sphere of radius comparable with a few mean free path actually contributes to the volume integral, due to the exponential term  $e^{-r/l_f}$ . Here  $\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i}_x + \sin \theta \sin \phi \mathbf{i}_y + \cos \theta \mathbf{i}_z$  and  $\mathbf{r} = r\mathbf{i}_r$ .

We note that, as derivatives are evaluated at a fixed point, each term is the projection of a constant vector along  $\mathbf{i}_r$ . Then, in the angular average:

- a single projection averages to zero due to the angular oscillating term:

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta [\mathbf{i}_r \cdot \mathbf{i}_\alpha] = 0 \quad (9)$$

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta [\mathbf{i}_r \cdot \nabla f_1(x, y, z)] = 0 \quad (10)$$

where  $f_1(x, y, z)$  is a scalar field and the gradient is evaluated at  $\mathbf{r} = 0$ .

- The product of two projection gives a finite value:

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta [\mathbf{i}_r \cdot \mathbf{i}_\alpha] [\mathbf{i}_r \cdot \nabla f_1(x, y, z)] = \frac{4}{3}\pi \frac{\partial f_1(x, y, z)}{\partial \alpha} \quad (11)$$

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta [\mathbf{i}_r \cdot \nabla f_1(x, y, z)] [\mathbf{i}_r \cdot \nabla f_2(x, y, z)] = \frac{4}{3}\pi \nabla f_1(x, y, z) \cdot \nabla f_2(x, y, z) \quad (12)$$

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta (\mathbf{i}_r \cdot \nabla)^2 f_1(x, y, z) = \frac{4}{3}\pi \nabla^2 f_1(x, y, z) \quad (13)$$

where  $f_2(x, y, z)$  is another scalar field and again the gradients is evaluated at  $\mathbf{r} = 0$ .

- The product of three projection gives again a null value:

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta [\mathbf{i}_r \cdot \mathbf{i}_\alpha] [(\mathbf{r} \cdot \nabla)^2 f_1(x, y, z)] = 0 \quad (14)$$

As an example, we evaluate Equation (11) in the case  $\alpha = x$ . Now  $[\mathbf{i}_r \cdot \mathbf{i}_\alpha] = \sin \theta \cos \phi$ , therefore

$$\begin{aligned} & \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \left( \sin \theta \cos \phi \frac{\partial f_1}{\partial x} + \sin \theta \sin \phi \frac{\partial f_1}{\partial y} + \cos \theta \frac{\partial f_1}{\partial z} \right) \sin \theta \cos \phi = \\ & \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \left( \sin^2 \theta \cos^2 \phi \frac{\partial f_1}{\partial x} \right) = \\ & \frac{\partial f_1}{\partial x} \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \sin^3 \theta d\theta = \frac{4}{3} \pi \frac{\partial f_1}{\partial x} \end{aligned}$$

The integration over the radius  $r$  is much simpler:

$$\int_0^\infty e^{-r/l_f} dr = l_f \quad (15a)$$

$$\int_0^\infty r e^{-r/l_f} dr = l_f^2 \quad (15b)$$

$$\int_0^\infty r^2 e^{-r/l_f} dr = 2l_f^3 \quad (15c)$$

The rate of momentum entering the volume element  $dV$  is then:

$$\frac{dp_\alpha}{dt} \simeq dV \gamma_0 \frac{B_v}{4} \left( \frac{3T_P^4 v_{P\alpha}}{l_f c} - 4T_P^3 \frac{\partial T_P}{\partial \alpha} + 8T_P^3 \frac{l_f}{c} \nabla T_P \cdot \nabla v_{P\alpha} + 12 \frac{l_f}{c} T_P^2 v_{P\alpha} (\nabla T_P)^2 + \frac{l_f}{c} T_P^4 \nabla^2 v_{P\alpha} \right) \quad (16)$$

In respect to the I order expansion in  $\mathbf{r}$  leading to the Equation (7) of the main text, now all contributions up to the first order in the mean free path  $l_f$  are considered. Further terms of the expansions (6) and (7) would add negligible contributions proportional to  $l_f^2$ . We have already discussed the first three elements of this equation, respectively the "thermal equilibrium", "phonon pressure" and "damping" term, and address now the relevance of the additional terms.

- The fourth term is a correction to the damping term, as it depends both on the temperature gradient and on the local velocity of the crystal. In the case of a silicon cantilever discussed in the main text, the correction is less than 10% over the considered range of relative gradient. We note that to derive this term we used the relation:

$$\nabla^2 (T_P^4) = 3T_P^2 (\nabla T_P)^2 + T_P^3 \nabla^2 T_P = 3T_P^2 (\nabla T_P)^2 \quad (17)$$

where  $\nabla^2 T_P = 0$  because  $T_P$  is a solution of the stationary heat equation with no source term.

- The fifth term is a correction to the thermal equilibrium term, therefore the associated loss angle concurs to determine the background loss. It can augment or reduce the overall damping of the resonator, depending on the modal shape and on the Grüneisen parameter  $\gamma$  of the lattice. Lacking a the dependence from the gradient, we believe that it is not reasonable to single out this phononic contribution among the other dissipative mechanism. As a consistency check, we only note that for the silicon cantilever discussed in the main text the expected contribution is a few units  $\times 10^{-6}$ , not far from the values measured in similar systems [4]. This term is consistent also with the extremely low loss factor measured in resonators exploiting torsional modes, such as the Double Paddle Oscillator [5]. In fact in this case the expected damping from phonon transport is orders of magnitude smaller due to the symmetry of the modal shapes.

## VOLUME INTEGRAL: MEDIUM TEMPERATURE APPROXIMATION FOR SILICON

From Equation (4), in the medium temperature case, we write the rate of momentum entering the volume element  $dV$  as:

$$\frac{d\mathbf{p}}{dt} = dV \frac{3}{4\pi c l_f^2} \int_{V_B} d\sigma F_{MT}(T_Q) \frac{1}{r^2} e^{-r/l_f} (-c \mathbf{i}_r + \mathbf{v}_Q) \quad (18)$$

with  $F_{MT}(T_Q)$  given by Equation (3). In this case it is straightforward to show that, to the first order in  $l_f$ , the contributions to the rate of momentum entering the volume element  $dV$  are:

- Thermal equilibrium term

$$dV \left( \frac{3v_{p\alpha}}{l_f c} + \frac{l_f}{c} \nabla^2 v_{p\alpha} \right) (F_0 + F_1 T_P + F_2 T_P^2) \quad (19)$$

- Phonon pressure term

$$- dV \frac{\partial T_P}{\partial \alpha} (F_1 + 2F_2 T_P) \quad (20)$$

- Damping term

$$dV \frac{2l_f}{c} (F_1 + 2F_2 T_P) \nabla T_P \cdot \nabla v_{p\alpha} + dV \frac{2l_f}{c} F_2 v_{p\alpha} (\nabla T_P)^2 \quad (21)$$

These local contributions were integrated over the time and over the volume of the cantilevered beam to obtain the total loss angle at 100 K and 150 K reported in Figure 2 of the main text.

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